2) pour $x \in \left[0, \frac{\pi}{4}\right]$: $\cos 2x \leq 1$

$x^n.\cos 2x \leq x^n$

$\Rightarrow \int_0^{\frac{\pi}{4}} x^n.\cos 2x\, dx \leq \int_0^{\frac{\pi}{4}} x^n\, dx$

$\Rightarrow U_n \leq \int_0^{\frac{\pi}{4}} x^n\, dx$

3) $x^n.\cos 2x \geq 0$ pour $x \in \left[0, \frac{\pi}{4}\right]$

$\Rightarrow \int_0^{\frac{\pi}{4}} x^n.\cos 2x\, dx \geq 0 \Rightarrow U_n \geq 0$

* $0 \leq U_n \leq \int_0^{\frac{\pi}{4}} x^n\, dx$

$\Rightarrow 0 \leq U_n \leq \left[\frac{x^{n+1}}{n+1}\right]_0^{\frac{\pi}{4}} \Rightarrow 0 \leq U_n \leq \frac{1}{n+1}(\frac{\pi}{4})^{n+1}$

$\lim_{n\to+\infty} \frac{1}{n+1}(\frac{\pi}{4})^{n+1} = 0$

d'où $\lim_{n\to+\infty} U_n = 0$

4) a) * $U_0 = \int_0^{\frac{\pi}{4}} \cos 2x\, dx = \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$U_0 = \frac{1}{2}$

* $U_1 = \int_0^{\frac{\pi}{4}} x.\cos 2x\, dx$

intégrant par parties, on pose :

$$\begin{cases} U'(x)=\cos 2x \xrightarrow{P} U(x) = \frac{1}{2}\sin 2x \\ V(x) = x \xrightarrow{d} V'(x) = 1 \end{cases}$$

$U_1 = \left[\frac{1}{2}x\sin 2x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{1}{2}\sin 2x dx$

$U_1 = \frac{\pi}{8} + \left[\frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{1}{4}$

b) $U_{n+2} = \int_0^{\frac{\pi}{4}} x^{n+2}.\cos 2x dx$

intégration par parties, on pose :

$$\begin{cases} U'(x)=\cos 2x \xrightarrow{P} U(x) = \frac{1}{2}\sin 2x \\ V(x) = x^{n+2} \xrightarrow{d} V'(x) = (n+2)x^{n+1} \end{cases}$$

$U_{n+2} = \left[\frac{1}{2}x^{n+2}.\sin 2x\right]_0^{\frac{\pi}{4}} - \frac{(n+2)}{2}.\int_0^{\frac{\pi}{4}} x^{n+1}.\sin 2x dx$

$U_{n+2} = \frac{1}{2}(\frac{\pi}{4})^{n+2} - \frac{(n+2)}{2}.\int_0^{\frac{\pi}{4}} x^{n+1}.\sin 2x dx$ (1)

faisons une 2éme intégration par parties :

$$\begin{cases} U'(x)=\sin 2x \xrightarrow{P} U(x) = -\frac{1}{2}\cos 2x \\ V(x) = x^{n+1} \xrightarrow{d} V'(x) = (n+1)x^n \end{cases}$$

$\int_0^{\frac{\pi}{4}} x^{n+1}.\sin 2x dx = \left[\frac{-x^{n+1}}{2}\cos 2x\right]_0^{\frac{\pi}{4}} + \frac{(n+1)}{2}\int_0^{\frac{\pi}{4}} x^n.\cos 2x$

$\int_0^{\frac{\pi}{4}} x^{n+1}.\sin 2x dx = \frac{(n+1)}{2}.U_n$ (2)

(1) + (2) $\Rightarrow$

$U_{n+2} = \frac{1}{2}(\frac{\pi}{4})^{n+2} - \frac{(n+1)(n+2)}{4}.U_n$

c)

* $U_2 = \frac{1}{2}(\frac{\pi}{4})^2 - \frac{1}{2}U_0 = \frac{\pi^2}{32} - \frac{1}{4}$

* $U_3 = \frac{1}{2}(\frac{\pi}{4})^3 - \frac{3}{2}U_1 = \frac{1}{2}(\frac{\pi}{4})^3 - \frac{3}{2}(\frac{\pi}{8} - \frac{1}{4})$

$U_3 = \frac{1}{2}(\frac{\pi}{4})^3 - \frac{3\pi}{16} + \frac{3}{8}$

Qarray.tn -Aqra Blech- Qarray.tn